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Continuous-state branching processes in temporally varying environments

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1. Time-homogeneous branching processes

• Galton-Watson processes (GW-processes)

Given Z_0, we define

$$Z_n = \sum_{i=1}^{Z_{n-1}} \xi_{n-1}(i), \qquad \text{for } n \geq 1,$$

where $\{\xi_n(i) : n = 0, 1, ...; i = 1, 2...\}$ are i.i.d. random variables $\sim g$ (the generating function).

1. Time-homogeneous branching processes

• Continuous-state branching processes (CB-processes)

We call X a CB-process with branching mechanism ϕ if it has transition semigroup $(Q_t)_{t\geq 0}$ defined by

$$\int_0^\infty \mathrm{e}^{-\lambda y} Q_t(x, \mathrm{d}y) = \mathrm{e}^{-x v_t(\lambda)}, \qquad \lambda,\ t \geq 0, \tag{1}$$

where the cumulant semigroup $(v_t)_{t\geq 0}$ is determined by

$$v_t(\lambda) = \lambda - \int_0^t \phi(v_s(\lambda))\mathrm{d}s, \qquad \lambda,\ t \geq 0, \tag{2}$$

and ϕ has the representation

$$\phi(z) = bz + cz^2 + \int_0^\infty (\mathrm{e}^{-zu} - 1 + zu) m(\mathrm{d}u), \tag{3}$$

where $c \geq 0, b$ are constants and $(u \wedge u^2)m(\mathrm{d}u)$ is a finite measure on $(0,\ \infty)$.

1. Time-homogeneous branching processes

• **Continuous-state branching processes (CB-processes)**

• **Scaling limits**

CB-processes are weak limits of rescaled GW-processes. See, e.g. Li ('06/'11).

1. Time-homogeneous branching processes

- **Continuous-state branching processes (CB-processes)**
- **Scaling limits**
- **Stochastic equations**

$W(\mathrm{d}s, \mathrm{d}u)$: Gaussian white noise with intensity $\mathrm{d}s\mathrm{d}u$;

$\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$: compensated Poisson random measure with intensity $\mathrm{d}s m(dz)\mathrm{d}u$.

Theorem 1. *(Dawson and Li ('06/'12)) There is a pathwise unique positive (strong) solution to*

$$X_t = x - \int_0^t b X_{s-} \mathrm{d}s + \sqrt{2c} \int_0^t \int_0^{X_{s-}} W(\mathrm{d}s, \mathrm{d}u) + \int_0^t \int_0^\infty \int_0^{X_{s-}} z \tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \tag{4}$$

The solution $\{X(t)\}$ of (4) is a CB-process with initial value x and branching mechanism ϕ.

2. Time-inhomogeneous branching processes

• GW-processes in varying environments (GWVE-processess)

Given Z_0, we define

$$Z_n = \sum_{i=1}^{Z_{n-1}} \xi_{n-1}(i), \qquad \text{for } n \geq 1,$$

where $\{\xi_n(i) : i = 1, 2...\}$ are i.i.d. random variables $\sim g_n$ (the generating function) for $n \geq 0$.

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Corresponding Continuous-state Processes?

2. Time-inhomogeneous branching processes

Bansaye and Simatos ('15) established a sufficient condition for a sequence of GWVE-processes to converge weakly. They considered the following scaling.

$\{Z_{k,n} : n \geq 0\}$: $Z_{k,n} = \sum_{i=1}^{Z_{k,n-1}} \xi_{k,n-1}(i)$. $(\{\xi_{k,n}(i) : i = 1, 2...\} \sim \xi_{k,n})$

$\{\gamma_k\}$: $t \mapsto \gamma_k(t)$ is an increasing, càdlàg and onto function from $[0, \infty)$ to $\mathbb{N}_+$.

$\{X_k(t) : t \geq 0\}$: $X_k(t) = Z_{k,\gamma_k(t)}/k$.

$\{u_k(r, t, \lambda)\}$: $\mathrm{P}(\mathrm{e}^{-\lambda X_k(t)} | X_k(r) = x) = \exp\{-x u_k(r, t, \lambda)\}$.

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$\{u_k(r, t, \lambda)\}$: $\mathrm{P}(\mathrm{e}^{-\lambda X_k(t)} | X_k(r) = x) = \exp\{-x u_k(r, t, \lambda)\}$.

Define $\nu_{k,n}([x, \infty)) = k\mathrm{P}(\xi_{k,n} \geq kx + 1)$ and

$\alpha_k(t) = \sum_{n=0}^{\gamma_k(t)-1} \int \frac{x}{1+x^2} \nu_{k,n}(\mathrm{d}x)$,

$\beta_k(t) = \sum_{n=0}^{\gamma_k(t)-1} \int \frac{x^2}{2(1+x^2)} \nu_{k,n}(\mathrm{d}x)$,

$\nu_k((0, t] \times [x, \infty)) = \sum_{n=0}^{\gamma_k(t)-1} \nu_{k,n}([x, \infty))$.

2. Time-inhomogeneous branching processes

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$\{X_k(t) : t \geq 0\}$: $X_k(t) = Z_{k,\gamma_k(t)}/k$.

$\{u_k(r,t,\lambda)\}$: $\mathrm{P}(\mathrm{e}^{-\lambda X_k(t)}|X_k(r) = x) = \exp\{-x u_k(r,t,\lambda)\}$.

Then for $0 \leq r \leq t$,

$$u_k(r,t,\lambda) \approx \lambda + \int_r^t u_k(s,t,\lambda)\alpha_k(\mathrm{d}s) - \int_r^t (u_k(s,t,\lambda))^2 \beta_k(\mathrm{d}s)$$
$$- \int_r^t \int_0^\infty \Big(\mathrm{e}^{-z u_k(s,t,\lambda)} - 1 + \frac{z u_k(s,t,\lambda)}{1+z^2} - \frac{(z u_k(s,t,\lambda))^2}{2(1+z^2)}\Big)\nu_k(\mathrm{d}s\mathrm{d}z).$$

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$$- \int_r^t \int_0^\infty \left(\mathrm{e}^{-zu_k(s,t,\lambda)} - 1 + \frac{zu_k(s,t,\lambda)}{1+z^2} - \frac{(zu_k(s,t,\lambda))^2}{2(1+z^2)}\right)\nu_k(\mathrm{d}s\mathrm{d}z).$$

Bansaye and Simatos ('15) introduced some assumptions on $\nu_{k,n}$, α_k, β_k and ν_k. For example, there exists a càdlàg function of locally finite variation α such that

for $t \ge 0$, $\alpha_k(t) = \sum_{n=0}^{\gamma_k(t)-1} \int \frac{x}{1+x^2}\nu_{k,n}(\mathrm{d}x) \to \alpha(t)$ as $k \to \infty$;

for $t > 0$ satisfying $\Delta\alpha(t) \neq 0$, $\Delta\alpha_k(t) \to \Delta\alpha(t)$ as $k \to \infty$.

Similar with α, they assumed that there exists an increasing càdlàg function β and a positive measure ν on $(0,\infty)^2$ which can be the pointwise limits of the related sequences.

2. Time-inhomogeneous branching processes

For $0 \leq r \leq t$,

$$u_k(r,t,\lambda) \approx \lambda + \int_r^t u_k(s,t,\lambda)\alpha_k(\mathrm{d}s) - \int_r^t (u_k(s,t,\lambda))^2 \beta_k(\mathrm{d}s) \\ - \int_r^t \int_0^\infty \left(\mathrm{e}^{-zu_k(s,t,\lambda)} - 1 + \frac{zu_k(s,t,\lambda)}{1+z^2} - \frac{(zu_k(s,t,\lambda))^2}{2(1+z^2)}\right)\nu_k(\mathrm{d}s\mathrm{d}z).$$

Bansaye and Simatos ('15) also introduced a bottleneck point $\wp(t)$:

$$\wp(t) = \sup\{s \leq t : \liminf_{k\to\infty} \inf_{s\leq y\leq t} u_k(y,t,\lambda) = 0\}.$$

Under the assumptions on $\nu_{k,n}$, α_k, β_k and ν_k, they characterized the behavior of the limit process on $[\wp(t), t]$.

2. Time-inhomogeneous branching processes

For $0 \le r \le t$,

$$u_k(r,t,\lambda) \approx \lambda + \int_r^t u_k(s,t,\lambda)\alpha_k(\mathrm{d}s) - \int_r^t (u_k(s,t,\lambda))^2 \beta_k(\mathrm{d}s)$$
$$- \int_r^t \int_0^\infty \Big(\mathrm{e}^{-zu_k(s,t,\lambda)} - 1 + \frac{zu_k(s,t,\lambda)}{1+z^2} - \frac{(zu_k(s,t,\lambda))^2}{2(1+z^2)}\Big)\nu_k(\mathrm{d}s\mathrm{d}z).$$

- The cumulant semigroup $(u_{r,t})_{t \ge r}$

For $\lambda > 0$ and $r \in [\wp(t), t]$,

$$u_{r,t}(\lambda) = \lambda + \int_r^t u_{s,t}(\lambda)\alpha(\mathrm{d}s) - \int_r^t u_{s,t}(\lambda)^2 \tilde{\beta}(\mathrm{d}s)$$
$$- \int_r^t \int_0^\infty \Big(\mathrm{e}^{-zu_{s,t}(\lambda)} - 1 + \frac{zu_{s,t}(\lambda)}{1+z^2}\Big)\nu(\mathrm{d}s\mathrm{d}z), \qquad (5)$$

where $t \mapsto \tilde{\beta}(t) = \beta(t) - \int_0^t \int_0^\infty \frac{z^2}{1+z^2}\nu(\mathrm{d}s\mathrm{d}z)$ is continuous and increasing and α, β, ν depend on the sequence of rescaled processes.

2. Time-inhomogeneous branching processes

- The cumulant semigroup $(u_{r,t})_{t\geq r}$

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$$u_{r,t}(\lambda) \;=\; \lambda + \int_r^t u_{s,t}(\lambda)\alpha(\mathrm{d}s) - \int_r^t u_{s,t}(\lambda)^2\tilde{\beta}(\mathrm{d}s) - \int_r^t\int_0^\infty\Big(\mathrm{e}^{-zu_{s,t}(\lambda)} - 1 + \frac{zu_{s,t}(\lambda)}{1+z^2}\Big)\nu(\mathrm{d}s\mathrm{d}z).$$

- The convergence of the rescaled processes

Suppose that $x_k \to x$, $(X_k(s) : r \leq s \leq t)$ under $\mathbf{P}(\cdot|X_k(r) = x_k)$ is tight for any $r \in [0, t]$. Bansaye and Simatos ('15) showed that the weak limit process of $(X_k(s) : r \leq s \leq t)$ for $r \in [\wp(t), t]$ has the cumulant semigroup $(u_{r,t})_{t\geq r}$.

2. Time-inhomogeneous branching processes

- The cumulant semigroup $(u_{r,t})_{t\geq r}$

For $\lambda > 0$ and $r \in [\wp(t), t]$,

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- The convergence of the rescaled processes

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Existence of such process?

2. Time-inhomogeneous branching processes

• **Integral evolution equation** (first moment condition)

Suppose that b is a càdlàg function of locally finite variation, c is an increasing continuous function, m is a σ-finite measure on $(0,\infty)^2$ satisfying $\int_0^t\int_0^\infty(z\wedge z^2)m(\mathrm{d}s,\mathrm{d}z)<\infty$ for $t\geq 0$.

In addition for $t>0$, $\Delta b(t)+\int_0^\infty zm(\{t\},\mathrm{d}z)\leq 1$.

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In addition for $t>0$, $\Delta b(t)+\int_0^\infty zm(\{t\},\mathrm{d}z)\leq 1$.

Theorem 2. *For every* $t,\lambda\geq 0$, *there is a unique bounded positive solution* $r\in[0,t]\mapsto v_{r,t}(\lambda)$ *to*

$$\begin{aligned} v_{r,t}(\lambda) \;=\; & \lambda-\int_r^t v_{s,t}(\lambda)b(\mathrm{d}s)-\int_r^t v_{s,t}(\lambda)^2 c(\mathrm{d}s) \\ & -\int_r^t\int_0^\infty\Big(\mathrm{e}^{-v_{s,t}(\lambda)z}-1+v_{s,t}(\lambda)z\Big)m(\mathrm{d}s,\mathrm{d}z). \end{aligned}$$

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• **Integral evolution equation** (first moment condition)

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In addition for $t>0$, $\Delta b(t)+\int_0^\infty zm(\{t\},\mathrm{d}z)\leq 1$.

Theorem 2. *For every $t,\lambda\geq 0$, there is a unique bounded positive solution $r\in[0,t]\mapsto v_{r,t}(\lambda)$ to*

$$\begin{aligned} v_{r,t}(\lambda) \;=\; \lambda &- \int_r^t v_{s,t}(\lambda)b(\mathrm{d}s) - \int_r^t v_{s,t}(\lambda)^2 c(\mathrm{d}s) \\ &- \int_r^t\int_0^\infty \Big(\mathrm{e}^{-v_{s,t}(\lambda)z} - 1 + v_{s,t}(\lambda)z\Big)m(\mathrm{d}s,\mathrm{d}z). \end{aligned} \tag{6}$$

If $b(\mathrm{d}s)=b\,\mathrm{d}s$, $c(\mathrm{d}s)=c\,\mathrm{d}s$ and $m(\mathrm{d}s,\mathrm{d}z)=\mathrm{d}s\,m(\mathrm{d}z)$, then (6) $\Rightarrow$ (2).

2. Time-inhomogeneous branching processes

• **Integral evolution equation** (first moment condition)

Suppose that b is a càdlàg function of locally finite variation, c is an increasing continuous function, m is a σ-finite measure on $(0,\infty)^2$ satisfying $\int_0^t \int_0^\infty (z \wedge z^2) m(\mathrm{d}s, \mathrm{d}z) < \infty$ for $t \geq 0$.

In addition for $t > 0$, $\Delta b(t) + \int_0^\infty z m(\{t\}, \mathrm{d}z) \leq 1$.

Theorem 2. *For every $t, \lambda \geq 0$, there is a unique bounded positive solution $r \in [0,t] \mapsto v_{r,t}(\lambda)$ to*

$$\begin{aligned} v_{r,t}(\lambda) \ = \ & \lambda - \int_r^t v_{s,t}(\lambda) b(\mathrm{d}s) - \int_r^t v_{s,t}(\lambda)^2 c(\mathrm{d}s) \\ & - \int_r^t \int_0^\infty \Big(\mathrm{e}^{-v_{s,t}(\lambda) z} - 1 + v_{s,t}(\lambda) z \Big) m(\mathrm{d}s, \mathrm{d}z). \end{aligned}$$

The solution satisfies the semigroup property

$$v_{r,t}(\lambda) = v_{r,s} \circ v_{s,t}(\lambda) = v_{r,s}(v_{s,t}(\lambda)) \ \text{ for } \ \lambda \geq 0, t \geq s \geq r \geq 0.$$

2. Time-inhomogeneous branching processes

- **Integral evolution equation** (first moment condition)

Suppose that b is a càdlàg function of locally finite variation, c is an increasing continuous function, m is a σ-finite measure on $(0,\infty)^2$ satisfying $\int_0^t\int_0^\infty(z\wedge z^2)m(\mathrm{d}s,\mathrm{d}z)<\infty$ for $t\geq 0$.

In addition for $t>0$, $\Delta b(t)+\int_0^\infty zm(\{t\},\mathrm{d}z)\leq 1$.

Theorem 2. *For every $t,\lambda\geq 0$, there is a unique bounded positive solution $r\in[0,t]\mapsto v_{r,t}(\lambda)$ to*

$$v_{r,t}(\lambda) = \lambda-\int_r^t v_{s,t}(\lambda)b(\mathrm{d}s)-\int_r^t v_{s,t}(\lambda)^2c(\mathrm{d}s) -\int_r^t\int_0^\infty\Big(\mathrm{e}^{-v_{s,t}(\lambda)z}-1+v_{s,t}(\lambda)z\Big)m(\mathrm{d}s,\mathrm{d}z).$$

The solution has the Lévy-Kthintchine representation

$$v_{r,t}(\lambda)=h_{r,t}\lambda+\int_0^\infty(1-\mathrm{e}^{-\lambda y})l_{r,t}(\mathrm{d}y),\ \ t\geq r\geq 0,\lambda\geq 0.$$

2. Time-inhomogeneous branching processes

• CB-processes in temporally varying environments

We call X a CB-process in temporally varying environments if it has transition semigroup $(Q_{r,t})_{t\geq r\geq 0}$ defined by

$$\int_0^\infty \mathrm{e}^{-\lambda y} Q_{r,t}(x, \mathrm{d}y) = \mathrm{e}^{-x v_{r,t}(\lambda)}, \qquad \lambda,\ t \geq 0, \tag{7}$$

where the cumulant semigroup $(v_{r,t})_{t\geq 0}$ is determined by

$$v_{r,t}(\lambda) = \lambda - \phi(v_{\cdot,t}(\lambda))((r,t]), \tag{8}$$

and the branching mechanism ϕ has the representation: for $f \in B(\mathbb{R}_+)^+$ and $B \in \mathscr{B}(\mathbb{R}_+)$,

$$\begin{aligned}\phi(f)(B) \ &= \ \int_B f(s) b(\mathrm{d}s) + \int_B f^2(s) c(\mathrm{d}s) \\ &\quad + \int_B \int_0^\infty \left(\mathrm{e}^{-f(s)z} - 1 + f(s)z\right) m(\mathrm{d}s, \mathrm{d}z),\end{aligned} \tag{9}$$

where b, c, m satisfy the preceding conditions.

2. Time-inhomogeneous branching processes

- **CB-processes in temporally varying environments**

- **Stochastic equations**

$$
\begin{aligned}
X_t \;=\; x - \int_0^t bX_{s-}\mathrm{d}s + \sqrt{2c}\int_0^t\int_0^{X_{s-}} W(\mathrm{d}s,\mathrm{d}u) \\
+ \int_0^t\int_0^\infty\int_0^{X_{s-}} z\tilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).
\end{aligned}
$$

2. Time-inhomogeneous branching processes

- **CB-processes in temporally varying environments**

- **Stochastic equations**

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$W(\mathrm{d}s,\mathrm{d}u)$: time-space white noise with intensity $2c(\mathrm{d}s)\mathrm{d}u$;

$\tilde{M}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$: compensated Poisson random measure with intensity $m(\mathrm{d}s,\mathrm{d}z)\mathrm{d}u$.

The noises are independent.

2. Time-inhomogeneous branching processes

- **CB-processes in temporally varying environments**

- **Stochastic equations**

Under some condition, we can prove that there is a unique positive strong solution to

$$\begin{aligned} X_t \;=\; x - \int_0^t X_{s-} b(\mathrm{d}s) + \int_0^t \int_0^{X_{s-}} W(\mathrm{d}s, \mathrm{d}u) \\ + \int_0^t \int_0^\infty \int_0^{X_{s-}} z \tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \end{aligned} \tag{10}$$

and the solution is a CB-processes in temporally varying environments with branching mechanism ϕ.

2. Time-inhomogeneous branching processes

- **CB-processes in temporally varying environments**

- **Stochastic equations**

Define $J_m = \{t > 0 : m(\{t\} \times (0, \infty)) > 0\}$ (non-random) and $J_m^c = (0, \infty) \setminus J_m$.

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Define $J_m = \{t > 0 : m(\{t\} \times (0,\infty)) > 0\}$ (non-random) and $J_m^c = (0,\infty)\setminus J_m$.

$$\begin{aligned}&\int_0^t \int_0^\infty \int_0^{X_{s-}} z\tilde{M}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) \\ &= \int_0^t \int_0^\infty \int_0^{X_{s-}} z\tilde{M}^c(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \sum_{s\le t, s\in J_m} \xi_s,\end{aligned}$$

where $\tilde{M}^c(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) = 1_{J_m^c}(s)\tilde{M}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$ and

$$\begin{aligned}\xi_s &= \int_0^\infty \int_0^{X_{s-}} z\tilde{M}(\{s\},\mathrm{d}z,\mathrm{d}u) \\ &= \int_0^\infty \int_0^{X_{s-}} zM(\{s\},\mathrm{d}z,\mathrm{d}u) - X_{s-}\int_0^\infty zm(\{s\},\mathrm{d}z), \qquad \text{for } s \in J_m.\end{aligned}$$

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$Thanks!$